An Algebraic Hierarchical Interpolative Factorization Algorithm for Ill-Conditionned Linear Systems

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Introduction

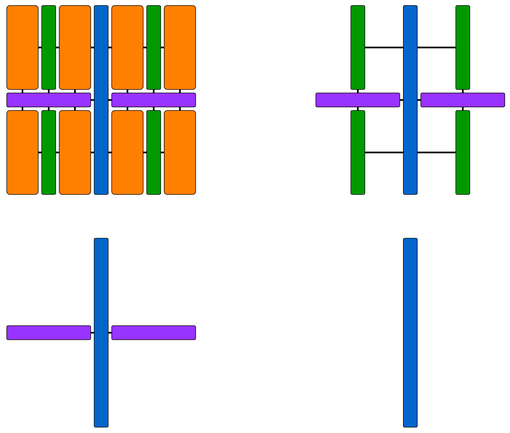
Direct methods (Sparse) LU + Pre-Ordering (ND)
- Very Robust
- Very Accurate
- Generic
- (Very) Costly

Incomplete Factorizations Incomplete LU / Sparsification methods / ...
- Tunable Accuracy
- Tunable Cost
- (Fairly) Generic
- Direct method or Preconditioner

Iterative methods CG / GMRES + Custom Preco
- Cheap
- (Very) Specific
- Complex Convergence
- Need Domain knowledge

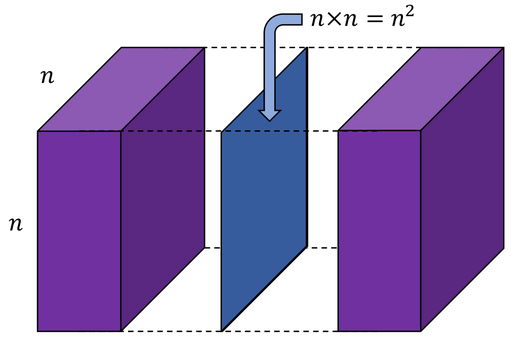
Nested Dissection

By eliminating disconnected partitions (interiors, separated by separators), nested dissection preserves sparsity.



Problem in 3D Meshes

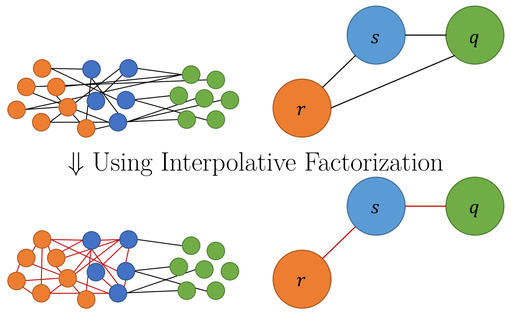
The cost is directly related to the size of the top-level separator. In 3D problems of size $N = n^3$, the top-level separator is of size $n^2 = N^{2/3}$, and its elimination costs $O\left((N^{2/3})^3\right) = O(N^2)$.



Interpolative Factorization

$$A_{qp}\Pi \approx A_{qs}\begin{bmatrix} I & T_{sr} \end{bmatrix} + O(\varepsilon) \text{ where } p = r \cup s$$

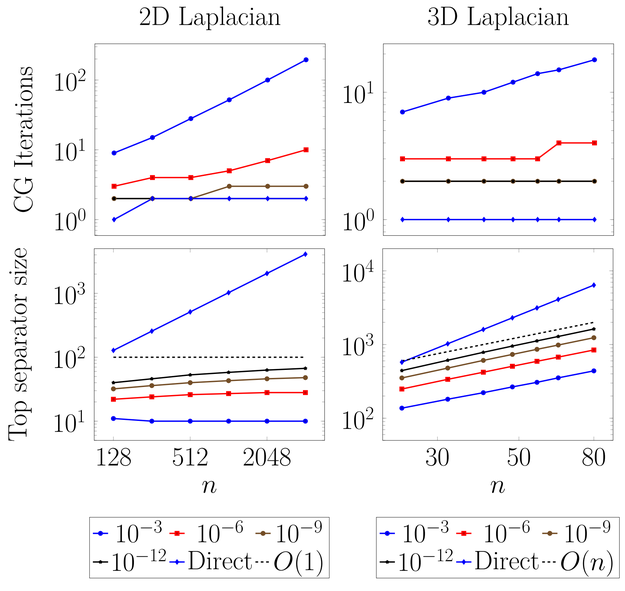
Sparse Elimination



General Algorithm

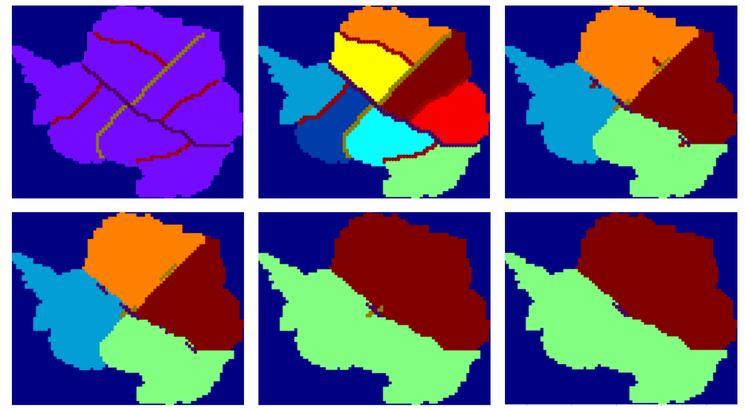
- Run a L-levels recursive *bisection* to build partitions
- Infer separators at level ℓ from the partitions at level ℓ
- For level $\ell = 1, \dots, L$
  - Eliminate interiors (usual ND) at level ℓ
  - Sparsify interfaces (partition at level $\ell \cap$ separators) at *all levels*

Scalings



Application on Irregular Meshes

We then apply this on all the interfaces and start eliminating the top-separator *from the beginning*, while introducing *no new edges*.



Res. [km]	N	ε	CG	Fwd err.	Bwd err.	s_{top}
64	63126	10^{-3}	17	$6.6 \cdot 10^{-6}$	$1.2 \cdot 10^{-2}$	25
		10^{-6}	5	$1.9 \cdot 10^{-7}$	$6.1 \cdot 10^{-4}$	83
		10^{-9}	3	$7.7 \cdot 10^{-9}$	$3.5 \cdot 10^{-6}$	164
		10^{-12}	2	$1.4 \cdot 10^{-9}$	$1.1 \cdot 10^{-8}$	232
32	245646	10^{-3}	33	$2.6 \cdot 10^{-1}$	$2.3 \cdot 10^{-2}$	17
		10^{-6}	15	$1.7 \cdot 10^{-1}$	$1.55 \cdot 10^{-2}$	77
		10^{-9}	4	$3.7 \cdot 10^{-5}$	$6.7 \cdot 10^{-6}$	177
		10^{-12}	2	$1.6 \cdot 10^{-8}$	$4.8 \cdot 10^{-9}$	267
16	969642	10^{-3}	73	1.2	$2.4 \cdot 10^{-2}$	18
		10^{-6}	16	$1.0 \cdot 10^{-1}$	$8.9 \cdot 10^{-3}$	80
		10^{-9}	4	$6.0 \cdot 10^{-5}$	$6.3 \cdot 10^{-6}$	170
		10^{-12}	2	$1.7 \cdot 10^{-8}$	$7.3 \cdot 10^{-9}$	250

References

- Kenneth Ho and Lexing Ying. "Hierarchical interpolative factorization for elliptic operators: differential equations." Communications on Pure and Applied Mathematics 69.8 (2016): 1415-1451
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